Kinematical Element Method for 3-D-Problems in Geomechanics

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Introduction

The few solutions which exist until now for 3-D-Problems in the theory of plasticity in combination with Coulomb's failure criterion are either based on very crude assumptions or are valid for very special problems only. The solutions of Hovland¹ and Herzog² for the slope problem disregard the shear forces in the vertical sections and the solutions of Hoek/Bray³ and Jinnan⁴ are applicable for simple rock wedges only. The author now presents an extension to three dimensions of the Kinematical Element Method (KEM) developed so far only for two dimensions, Gußmann⁴,⁵.

As the geometry is described consequently with respect to the coordinates of the nodal points, the formulas for the solution are fairly simple and general at the same time. The kinematics can be determined by the solution of a system of linear equations and the statics by another one. The variation of the geometry will be done by different optimization procedures. A first example for the field of passive earth pressure shows interesting results.

Geometry

For all Kinematical Methods an appropriate failure mechanism has to be designed. In the KEM the chosen failure mechanism consists of different rigid elements with plane boundaries, which are allowed only to translatory (virtual) movements, but not to rotations. In principle all convex polyhedrons could be considered as appropriate element types. Out of the family of polyhedrons the author presents in this paper just 4 types, the tetrahedron, the quadrilateral pyramid, the pentahedron and the hexahedron, (Figure 1). As each polyhedron can be subdivided into one or more tetrahedrons, the description of the elements can be reduced to the
description of the tetrahedron. The latter is uniquely defined by its four nodes $P_1$ to $P_4$ with their coordinates $(x_i, y_i, z_i)$ relative to a fix cartesian coordinate system. Some necessary formulas for area and cosines of the outward normal vector of a plane, volume etc. are given in the Appendix. As for a general problem the global numbering of the nodes and planes should be somehow arbitrary, the Author proposes a second local numbering order for each element with respect to the numbering order of its prototype.

![Tetrahedron](image1)

![Quadrilateral Pyramid](image2)

![Pentahedron](image3)

![Hexahedron](image4)

Figure 1. Four different 3D Elements

**Kinematics**

The design of a kinematically admissible failure mechanism is not as easy as in two dimensions. A design principle for constructing different admissible failure mechanisms can not be given so far. Instead of this a rule will be given to check a designed mechanism of different elements to be kinematically admissible or not.

Let us consider for the moment such an admissible mechanism and out of this mechanism an arbitrary element $f$.

Due to an induced, prescribed (virtual) movement of the moveable boundary, all the rigid elements of the mechanism will move too. As they can only slide along their boundary planes, no rotations are possible, but only translation. So each movement is just defined by the three cartesian
components of the absolute displacement vector of an arbitrary point of
the rigid element. Therefore no distinction has to be made for points
within the element and the notation

\[ \{v^f\} = \{v^f_x, v^f_y, v^f_z\} \]  \hspace{1cm} (1)

seems reasonable.

If just three adjacent elements c, d and e of element f with one common
point (the point of intersection of the three boundary planes in the
initial state) will be moved arbitrarily and independently (but only
translatorical) the new location of element f is explicitly defined.

According to Figures 2 and 3, which represent the kinematics of the
2-D-Problem for simplicity, the mathematical extension to three dimensions
can be derived.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figures.png}
\caption{Figure 2. Kinematics for a 2D Problem}
\end{figure}

Figure 2a shows the initial state of a passive earth pressure problem,
Figure 2b the displaced elements with emphasis to the location of element
f (now \( f_1 \)). and figure 2c shows two different positions of f, namely
\( f_1' \) without dilatancy, and \( f_1'' \), taking dilatancy into account.
Let denote

\[ v_{c/f} = v_f - v_c \]  \hspace{1cm} the vector of the relative displacement of element f
relative to element c

\[ v_{c/f} \]  \hspace{1cm} the normal component of this vector, referring to element c
\( \gamma_{c/f} \) = the angle of dilatancy, acting in the boundary between these two elements

\( n_{f/c} \) = the (outward) normal vector of the common plane, referring to element c

\( \alpha_{f/c}, \beta_{f/c}, \gamma_{f/c} \) = the angles between the outward normal and the axes \( x, y, z \)

then the fundamental equation for the kinematics will be, see Figure 3.

![Figure 3. Kinematics for Element f relative to Element c](image)

\[
V_{c/f, x} \cos \alpha_{f/c} + V_{c/f, y} \cos \beta_{f/c} + V_{c/f, z} \cos \gamma_{f/c} = V_{c/f}^{N c}
\]  
(2)

The normal component of the right hand side of equation (2) can be expressed by

\[
V_{c/f}^{N c} = (V_{c/f, x} + V_{c/f, y} + V_{c/f, z}) \cdot \sin \gamma_{c/f}
\]  
(3)

As the quadratic terms bring in some trouble for the numerical solution we
will simplify to nondilatant material or \( \psi_{c/f} = 0 \) and get

\[
1_{f/c} v_{c/f,x} + m_{f/c} v_{c/f,y} + n_{f/c} v_{c/f,z} = 0
\]  

(2a)

where \( l, m \) and \( n \) are abbreviations for the direction cosines. To be sure, that we really calculate the cosines of the outward normal of each element, a local coordinate system, located at any interior point of the element seems advisable. According to the definition of the vector \( v_{c/f} \) we derive for the components

\[
v_{c/f,x} = v^f_x - v^c_x
\]

\[
v_{c/f,y} = v^f_y - v^c_y
\]

\[
v_{c/f,z} = v^f_z - v^c_z
\]

(4)

together with (Figure 4):

\[
1_{c/f} = -1_{f/c}
\]

\[
m_{c/f} = -m_{f/c}
\]

\[
n_{c/f} = -n_{f/c}
\]

(5)

and get finally

\[
1_{c/f} v^f_x + m_{c/f} v^f_y + n_{c/f} v^f_z + 1_{f/c} v^c_x + m_{f/c} v^c_y + n_{f/c} v^c_z = 0
\]

(6.1)

The corresponding equations for the planes between the elements \( d \) and \( e \) relative to element \( f \) can be found by

\[
1_{d/f} v^f_x + m_{d/f} v^f_y + n_{d/f} v^f_z + 1_{f/d} v^d_x + m_{f/d} v^d_y + n_{f/d} v^d_z = 0
\]

(6.2)

\[
1_{e/f} v^f_x + m_{e/f} v^f_y + n_{e/f} v^f_z + 1_{f/e} v^e_x + m_{f/e} v^e_y + n_{f/e} v^e_z = 0
\]

(6.3)

If the boundary of an element \( f \) is rigid, the displacement of this "element" is zero.
\{ \mathbf{v} \} = \{ 0, 0, 0 \} \quad (7)

and therefore we call the whole rigid area an element 0. For the flexible boundary the induced displacements are known, they get the notation

\{ \mathbf{\hat{v}}_l \} = \{ \mathbf{\hat{v}}^f_l, \mathbf{\hat{v}}^r_l, \mathbf{\hat{v}}^l_l \} \quad (8)

If we assemble all elements of the failure mechanism we finally get the following system of linear equations

\{ K_v \} \{ \mathbf{v} \} + \{ \mathbf{\hat{v}}^N \} = 0 \quad (9)

where

\{ K_v \} is a non-symmetric matrix, which contains just the direction-cosines of the outward normals of the planes with 6 (or less) terms in each equation which are not zero,

\{ \mathbf{v} \} is the vector of the unknown (absolute) displacement values of the elements

\{ \mathbf{\hat{v}}^N \} is the vector of the normal components of the flexible boundary which cause the displacements.

In case of dilatancy we would have received a system of quadratic equations instead of the linear system. Once equation (9) is solved (with the elimination method of Gauss together with pivoting for example) the direction cosines of the relative displacement vectors can be calculated to

\begin{align*}
\bar{\mathbf{c}}/f &= \cos \bar{x} \mathbf{c}/f = \mathbf{v}_{c/f, x} / | \mathbf{v}_{c/f} | \\
\bar{\mathbf{m}}/f &= \cos \bar{y} \mathbf{c}/f = \mathbf{v}_{c/f, y} / | \mathbf{v}_{c/f} | \\
\bar{\mathbf{n}}/f &= \cos \bar{z} \mathbf{c}/f = \mathbf{v}_{c/f, z} / | \mathbf{v}_{c/f} | \\
| \mathbf{v}_{c/f} | &= (\mathbf{v}_{c/f, x}^2 + \mathbf{v}_{c/f, y}^2 + \mathbf{v}_{c/f, z}^2)^{1/2} \quad (10)
\end{align*}

and so on.

These direction-cosines will be used in the statics.
Statics

The force \( \{S\} \) which acts in the plane \( s \) onto the element \( f \) can be splitted up into a normal component \( \{N\} \) and a tangential component \( \{T\} \).

\[ \{S\} = \{N\} + \{T\} \quad (12) \]

The cartesian components of \( \{N\} \) can be seen from figure (5) to be

\[ N_{e/f,x} = -N_{e/f}^l_{e/f} \]
\[ N_{e/f,y} = -N_{e/f}^m_{e/f} \quad (13) \]
\[ N_{e/f,z} = -N_{e/f}^n_{e/f} \]

\[ \nabla T \parallel \nabla T \]
\[ \bar{n} \{\bar{e}, \bar{p}, \varphi\} \]

Figure 5. Forces acting on a Boundary Plane
As the tangential force \( T_{e/f} \) is directed opposite to the displacement-vector \( v_{e/f} \) - which is known from the kinematics - the cartesian components of \( T \) are

\[
T_{e/f,x} = -T_{e/f} \bar{e}_{e/f}
\]
\[
T_{e/f,y} = -T_{e/f} \bar{m}_{e/f}
\]
\[
T_{e/f,z} = -T_{e/f} \bar{n}_{e/f}
\]

(14)

The normal force \( N \)

\[
N = N' + U
\]

(15)

consists of normal effective force \( N' \) and neutral pore pressure force \( U \), the tangential shear force

\[
T = R + C
\]

(16)

consists of friction \( R \) and cohesion \( C \) in case of a Coulomb material. Together with

\[
N' = Q \cos \varphi \quad ; \quad R = Q \sin \varphi
\]
\[
C = c F \quad ; \quad U = u F
\]

(17)

where \( Q \) is the resultant force of \( N' \) and \( R' \), \( \varphi \) is the friction angle, \( C \) and \( U \) can be calculated from the area \( F \) and the mean values \( c \) and \( u \) and \( F \).

The cartesian components of force \( S \) can then be expressed as

\[
S_{e/f,x} = -\{Q_{e/f}(1_{e/f}\cos \varphi_{e/f} + \bar{e}_{e/f}\sin \varphi_{e/f}) + C_{e/f}\bar{e}_{e/f} + U_{e/f}\bar{e}_{e/f}\}
\]
\[
S_{e/f,y} = -\{Q_{e/f}(m_{e/f}\cos \varphi_{e/f} + \bar{m}_{e/f}\sin \varphi_{e/f}) + C_{e/f}\bar{m}_{e/f} + U_{e/f}\bar{m}_{e/f}\}
\]
\[
S_{e/f,z} = -\{Q_{e/f}(n_{e/f}\cos \varphi_{e/f} + \bar{n}_{e/f}\sin \varphi_{e/f}) + C_{e/f}\bar{n}_{e/f} + U_{e/f}\bar{n}_{e/f}\}
\]

(18)

The equilibrium equations for each elements require

\[
\sum_s S_{sx} = P_x = 0
\]
\[
\sum_s S_{sy} = P_y = 0
\]
\[
\sum_s S_{sz} = P_z = 0
\]

(19)
where the sum has to go over all boundary planes of the elements and where \( \{P\} \) contains the inertia forces, free surface loads or other known forces acting onto the elements.

Assembling all elements, the statics can be expressed by the following system of linear equations

\[
\{K_s\} \{Q\} + \{F\} = 0
\]  

(20)

where is

\( \{K_s\} \) a non-symmetric "friction"-matrix

\( \{Q\} \) the vector of the unknown forces \( Q_s \),

\( \{F\} \) the load vector, containing inertia forces, cohesion, pore pressure forces, surface loads etc.

Thus the statics can be solved with the same mathematical procedure as the kinematics.

Objective function and optimization

If \( f \) is a scalar, which represents the external work of the flexible boundary, \( f \) can be expressed as

\[
f = \sum \hat{v}^{f\ell}_x S_x^{f\ell} + \sum \hat{v}^{f\ell}_y S_y^{f\ell} + \sum \hat{v}^{f\ell}_z S_z^{f\ell} = \{\hat{v}^{f\ell}\}^t \{S^{f\ell}\}
\]  

(21)

To find the decisive geometry of the chosen failure mechanism for a certain boundary value problem, the objective function \( f \) has to be minimized with respect to the kinematical variables. For soil mechanics problems we chose the "free" coordinates \( (X) \) of the problem to be these kinematical variables -the other coordinates are either fix or depend on the surface geometry etc. and can be calculated - and the optimization problem reads

\[
f^* = f_{\min} (X^*) \ ; \ Q_j \geq 0 \ ; \ F_k \geq 0 \ ; \ V_i \geq 0 \ .
\]  

(22)

where the following nonlinear inequality constraints have to be considered: all forces \( Q_j \) must be pressure forces, all areas \( F_k \) of the boundary planes of each element and all volumes \( V_i \) must be positive. As the scalar function \( f \) is not always convex according to figure 6, the optimization problem remains a complicated part in the whole method.
Figure 6. Illustration of some Difficulties in Minimization

Together with the experiences of the 2-D-Problem the Author proposes the following steps:

- Design of an initial failure mechanism where not a single constraint is violated. If the problem is complicated this can be achieved by working through just a rough grid of the kinematical variables,

- applying the simplex-algorithm of Nealder/Mead\(^*\), or the complex-algorithm of Box\(^*\),

- or by the evolution-algorithm of Rechenberg\(^*\), given in a program of Schwefel\(^*\),

- refining the geometry by Davidon's quasi-Newton method, given in a program of Davidon/Nazareth\(^*\).
Application for passive earth pressure

In case of a cohesionless soil the horizontal component of the 2D-earth pressure is usually expressed by

\[ E_{ph} = \gamma \frac{H^2}{2} K_{ph} \]

(23)

where \( K_{ph} \) is the two-dimensional bearing-capacity factor of the horizontal passive earth pressure.

According to Weißenbach\textsuperscript{3,4}, the three-dimensional earth pressure can be calculated by

\[ E_{ph}^* = \begin{bmatrix} 1 & 0.3H \\ 0.3H & B \\ \end{bmatrix} \times \begin{bmatrix} 0.3H \\ B + 0.6H \tan \phi \end{bmatrix} \cdot E_{ph} \]

\[ \begin{cases} \frac{B}{H} \leq 0.3 \\ \frac{B}{H} \geq 0.3 \end{cases} \]

This formulation is supported by several experimental findings, Weißenbach\textsuperscript{4}. By using the following two different, but simple formulations

\[ E_{ph}^* = \gamma \frac{H^2}{2} K_{ph}^* B \]

(25)

\[ E_{ph}^* = \nu B E_{ph} \]

(26)

where \( K_{ph}^* \) is a three-dimensional bearing-capacity-factor and \( \nu \) is a shape-factor,

\[ \nu = \frac{K_{ph}^*}{K_{ph}} \]

(27)

is obtained and together with eq.(24)

\[ \nu_W = \begin{bmatrix} 1 & 0.3H \\ 0.3H & 1 + 0.6H \tan \phi \end{bmatrix} \times \begin{bmatrix} \frac{B}{H} \\ \frac{B}{H} \end{bmatrix} \]

\[ \begin{cases} \frac{B}{H} \leq 0.3 \\ \frac{B}{H} \geq 0.3 \end{cases} \]

with the limit values

\[ \lim_{B \to 0} \nu_W = 1 \]

(28.1)

\[ \lim_{B \to \infty} \nu_W = \infty \]

(28.2)

(\( \nu_W \) denotes the shape-factor according to Weißenbach\textsuperscript{3}).
The simplest KEM failure mechanism for this problem consists of just three elements (or two elements in case of symmetry), shown in Figure 7. But before the shape factor can be computed from eq. (27), we have also to investigate the 2D-problem with the KEM. Figure 8 shows five different failure mechanisms of the plane problem. The $K_{ph}$ values range from 5.74 to 5.02 for a problem with a friction angle $\phi = 30^\circ$ and an assumed wall friction value $\delta = 2/3\phi$. The tables of Pregl, which are based on the numerical integration of the differential equations of Kötter, according to Sokolovsky's method, give a lower bound with $K_{ph} = 4.95$. It seems reasonable - yet not necessary - to compare the simplest 3D-mechanism with the simplest 2D-mechanism yielding to $K_{ph} = 5.74$. Figure 9 shows the comparison with the plane problem and with Weißenbach's solution for different values $B/H$.

Figure 8. 2D-Earth Pressure Problem with KEM
Conclusions

The development of the 3D-Kinematical Element Method for limit load problems closes a gap in the field of soil and rock mechanics. Even the very simple failure mechanism of just two elements (in case of symmetry) for an earth pressure problem shows the effectiveness of the proposed method. The Author believes that the KEM will soon become a powerful tool for all geotechnical engineers.

References

13. A. Weissenbach, 'Der Erdwiderstand vor schmalen Druckflächen', Die Bau-
technik 39, 204-211 (1962).
14. A. Weissenbach, 'Der Erdwiderstand vor schmalen Druckflächen', Mit-
teilungen der Hannoverschen Versuchsanstalt für Grund- und Wasserbau.
15. O. Pregl and R. Kristof, 'Beiwerte für den passiven Erddruck', Geo-
16. F. Köpper, 'Die Bestimmung des Druckes an gekrümmten Gleitflächen',

Appendix

Area of a triangle in space:

\[ A = \frac{1}{2} \sqrt{\left( x_1 y_2 z_3 + x_2 y_3 z_1 + x_3 y_1 z_2 - x_2 y_2 z_3 - x_1 y_3 z_1 - x_3 y_1 z_2 \right)^2} \]

Volume of a tetrahedron:

\[ V = \frac{1}{6} \left| \begin{array}{ccc}
  x_1 & y_1 & z_1 \\
  x_2 & y_2 & z_2 \\
  x_3 & y_3 & z_3 \\
  x_4 & y_4 & z_4 \\
\end{array} \right| \]

Vector length:

\[ d = \sqrt{\left( x_2 - x_1 \right)^2 + \left( y_2 - y_1 \right)^2 + \left( z_2 - z_1 \right)^2} \]

Equation of a plane:

\[ \begin{array}{cccc}
  x & y & z & 1 \\
  x_1 & y_1 & z_1 & 1 \\
  x_2 & y_2 & z_2 & 1 \\
  x_3 & y_3 & z_3 & 1 \\
\end{array} = 0 \]

or

\[ A x + B y + C z + D = 0 \]

or

\[ 1 x + m y + n z + p = 0 \]

with the direction cosines:

\[ l = \frac{A}{W} \quad m = \frac{B}{W} \quad n = \frac{C}{W} \]

and with:

\[ p = - \frac{D}{W} \quad W = \text{sign}(-D) \left( A^2 + B^2 + C^2 \right)^{0.5} \]