Effective KEM solutions for the limit load and the slope stability problem

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Abstract. The matrix formulation for the limit load problem in soil mechanics can be solved by the KEM effectively. The solution of the slope stability problem is more complicated in principle, but together with an effective iteration technique the solution of the implicit equation system can be simplified to the recursive solution of just a quadratic equation.

1. Introduction

Matrix formulations for different mechanical problems in general have several advantages in comparison to their more analytical formulated solutions, especially

- a more direct derivation of compact formulas,
- an easier understanding of the solution,
- an easier program code,
- the equations are a good base for further extensions.

This paper may be understood either as a summary of KEM publications together with new solution techniques or as a matrix formulation of another recent paper, Gussmann [1], which is without matrices and more related to the general slice methods for slope stability problems.

2. Theory

The Kinematical Element Method (KEM) was presented first by Gussmann [2] and developed further in different publications. So the derivation of the new theory items may be short and compact. Because of some slight modifications concerning mainly the notation, the derivation of the formulas is shifted to the appendix, which is not always referenced in the main part of the paper.

2.1. Kinematics (no dilatancy). Together with the element nodes the geometry of an element, which is assumed to be rigid, can be described by the outward normal of a boundary according to figure 1. If we accept only sliding of elements within the boundaries and no dilatancy or penetration of elements, we can derive the equations for the virtual displacements from the condition, that the normal component of the relative displacement between two adjacent elements must vanish.

The linear set equations for a kinematical system of elements can be summarized by

\[
K_v \mathbf{v} + \dot{\mathbf{v}}_n = 0 \Rightarrow \mathbf{v} = -K_v^{-1} \dot{\mathbf{v}}_n = (K_n^T)^{-1} \dot{\mathbf{v}}_n \\
\mathbf{v}^T = \dot{\mathbf{v}}_n \left[ (K_n^T)^{-1} \right]^T = \dot{\mathbf{v}}_n^T K_n^{-1} \\
\mathbf{v}_l = \mathbf{v}_{0,l} + \dot{\mathbf{v}}_l = -K_l^T \mathbf{v} + \dot{\mathbf{v}}_l, \mathbf{v}_l^T = -\mathbf{v}^T K_l + \dot{\mathbf{v}}_l^T \\
\delta = \text{diag} \left[ \text{sign} \left( \mathbf{v}_l^T \right) \right]
\]
Figure 1: Element notations

Vector \( \mathbf{v} \) contains the cartesian components of the displacements and \( \mathbf{v}_t \) the tangential components of displacements in the boundaries, which are identical to the relative displacements between the elements. \( \mathbf{K}_n \) is a matrix containing the direction cosines of the outward normals of the element boundaries. For an effective solution avoiding \( \mathbf{K}_p^{-1} \) see appendix.

2.2. Statics. For the global system we can write the linear set of equilibrium equations together with the different physical forces. \( \mathbf{N}' \) is the vector of the yet unknown effective normal forces, \( \mathbf{U} \) the vector for the pore pressure forces, \( \mathbf{R} \) for the friction and \( \mathbf{C} \) for the cohesion forces.

\[
\mathbf{N} = \mathbf{N}' + \mathbf{U}, \mathbf{T} = \mathbf{R} + \mathbf{C}, \mathbf{R} = \mathbf{N}' \tan \varphi, \mathbf{S} = \mathbf{N} + \mathbf{T}
\]

(5)

\[
\mathbf{F}_p = - (\mathbf{P} + \mathbf{B} + \mathbf{A})
\]

(6)

where \( \mathbf{P}, \mathbf{B} \) and \( \mathbf{A} \) are vectors containing single, body and surface forces respectively.

\[
\begin{align*}
\mathbf{K}_n \mathbf{N} + \mathbf{K}_t \mathbf{T} &= \mathbf{F}_p \\
\mathbf{K}_n \left( \mathbf{N}' + \mathbf{U} \right) + \mathbf{K}_t \left( \mathbf{t}_n \mathbf{N}' + \mathbf{C} \right) &= \mathbf{F}_p \\
\left( \mathbf{K}_n + \mathbf{K}_t \mathbf{t}_n \right) \mathbf{N}' &= \mathbf{F}_p - \mathbf{K}_n \mathbf{U} - \mathbf{K}_t \mathbf{C} = \mathbf{F}_p + \mathbf{F}_U + \mathbf{F}_C = \mathbf{F} \\
\left( \mathbf{K}_n + \mathbf{K}_C \right) \mathbf{N}' &= \mathbf{F} \text{ or } \mathbf{K} \mathbf{N} = \mathbf{F} \Rightarrow \mathbf{N}' = \mathbf{K}^{-1} \mathbf{F}
\end{align*}
\]

(7) (8) (9) (10)
**Limit load problem and two effective solution techniques.** Instead of building the inverse \( K^{-1} \) we may derive better solutions.

In any case we can number elements and boundaries in such a way, that each element contains just 2 new unknown forces \( N' \). Then we decompose matrix \( K \) into a banded matrix \( K_B \) with a bandwidth of 2 and a sparse lower matrix \( K_\Delta \) beyond the diagonal according to

\[
K = K_B + K_\Delta \tag{11}
\]

We extend and solve as follows

\[
N' = K^{-1}F = K^{-1}K_B K_B^{-1}F \tag{12}
\]
\[
K_B^{-1} = DK_B^{adj} \tag{13}
\]
\[
N' = K^{-1}K_B DK_B^{adj}F = WDf = Wf_D = f^* \tag{14}
\]
\[
W = K^{-1}K_B, K_B = KW = K_BW + K_\Delta W \tag{15}
\]
\[
f = K_B^{adj}F, f_D = Df, B_\Delta = K_B^{-1}K_\Delta \tag{16}
\]
\[
W = (I + K_B^{-1}K_\Delta)^{-1} = (I + B_\Delta)^{-1} \tag{17}
\]

See appendix for the derivation of \( W \) and figure 2 for interpretation of \( \nu \).

\[
W = (I + B_\Delta)^{-1} = I + \sum_{i=1}^{\nu} (-1)^i B_\Delta^i, B_\Delta^{i+1} = 0, \nu = m + n - 2 \tag{18}
\]
\[
K^{-1} = WK_B^{-1} = WD\delta_B^{-1} \tag{19}
\]

This means we have to develop just \( W \) and \( K_B^{-1} = DK_B^{adj} \), where \( D \) is a diagonal matrix containing the inverse determinants of the single elements and \( K_B^{adj} \) is the adjugate of the elements, which provides a very efficient solution.

Besides this we can derive even a quicker solution, if we use the following forward update formulation.

\[
(K_B + K_\Delta) N' = F \tag{20}
\]
\[
N' = K_B^{-1}(F - K_\Delta N') \tag{21}
\]
\[
N' = f_D - B_\Delta N' = f^* \tag{22}
\]

Together with the transposed assigning vector \( r^T \) we get the limit load

\[
P_h = r^T N' = 0 \tag{23}
\]

In case of only one row of elements we just have \( P_h = N_{2n} \).

**Slope stability problem and effective iteration scheme.** Introducing factor \( F_S \) for the slope stability according to Fellenius [3] we decompose some but not all matrices again and get
m=3 rows, n=4 columns of elements
v =n+m-2=5 inner slide lines

Figure 2: System of elements with m=3 rows

\[
\begin{align*}
K_B &= K_{0,B} + \frac{1}{F_S} K_{1,B} \\
K_B^{-1} &= D K_{0,B}^{adj} D = D \left( K_{0,B}^{adj} + \frac{1}{F_S} K_{1,B}^{adj} \right) \\
D &= \text{diag} \left\{ \frac{1}{D_1}, \frac{1}{D_1}, \frac{1}{D_2}, \frac{1}{D_2}, \ldots, \frac{1}{D_n}, \frac{1}{D_n} \right\} \\
f_D &= D \left( K_{0,B}^{adj} + \frac{1}{F_S} K_{1,B}^{adj} \right) \left( F_0 + \frac{1}{F_S} F_1 \right) \\
f_D &= D f_0 + \frac{D}{F_S} f_1 + \frac{D}{F_S} f_2 = f_{0,D} + \frac{1}{F_S} f_{1,D} + \frac{1}{F_S} f_{2,D}
\end{align*}
\]

It must be remarked, that each element determinant $D_i$ is quadratic in terms of $F_S$ and therefore the above and the following equations are only valid within an iteration scheme, where convergency is supposed.

\[
\begin{align*}
N' &= W \left( f_{0,D} + \frac{1}{F_S} f_{1,D} + \frac{1}{F_S} f_{2,D} \right) = W_D \left( f_0 + \frac{1}{F_S} f_1 + \frac{1}{F_S} f_2 \right) \\
N' &= f'_0 + \frac{1}{F_S} f'_1 + \frac{1}{F_S} f'_2
\end{align*}
\]

If we use the update variant according to the solution for the limit load problem, we get

\[
\begin{align*}
N' &= \left( f_{0,D} + \frac{1}{F_S} f_{1,D} + \frac{1}{F_S} f_{2,D} \right) - B = \left( N_0 + \frac{1}{F_S} N'_1 + \frac{1}{F_S} N'_2 \right) \\
N' &= f'_0 + \frac{1}{F_S} f'_1 + \frac{1}{F_S} f'_2
\end{align*}
\]
To finish the solution we have to seek for that factor of safety, which lets vanish the resultant of the fictive limit load \( P_b = r^T N = 0 \). In general we may express this in a more physical way using the virtual work

\[
\hat{A} = \hat{v}_n^T \mathbf{N} = 0 \Rightarrow r = \hat{v}_n
\]

(33)

So we finally can derive a quadratic equation, which has to be solved recursively for \( F_S \)

\[
\begin{align}
\text{Norm} & = \hat{v}_n^T F^* \\
a_1 & = \hat{v}_n^T F^*/\text{Norm} \\
a_0 & = \hat{v}_n^T F^*/\text{Norm}
\end{align}
\]

\[
F_S^2 + a_1 F_S + a_0 = 0
\]

(37)

\[
F_S = -\frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0}
\]

(38)

The positive sign of the square root has to be choosen, because \( a_1 \) is negative.

The iteration scheme may be seen from the flow chart.

<table>
<thead>
<tr>
<th>0.</th>
<th>Compute: ( v, v_1, \delta, K_{0, 0}, K_{1, 0}, K_{0, 1}, K_{1, 1}, F_0, F_1 ) and set (for example): ( \varepsilon = 10^{-3}, k_{\text{max}} = 10 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Begin iteration: ( F_S := F_{S, 0}, k := k + 1 )</td>
</tr>
<tr>
<td>2.</td>
<td>Compute: ( D, B, f_0, f_1, f_2, f_3 )</td>
</tr>
<tr>
<td>3.</td>
<td>Solve for: ( f_0, f_1, f_2, f_3 ), compute: norm, ( a_1, a_0 ), solve the quadratic equation for ( F_S )</td>
</tr>
<tr>
<td>4.</td>
<td>If ( k \geq k_{\text{max}} ) : goto step 4. (no convergence)</td>
</tr>
<tr>
<td>5.</td>
<td>End iteration. If all ( \xi_i \geq 0 ) but those in ( P_b ) and (</td>
</tr>
<tr>
<td>6.</td>
<td>Change initial value ( F_{S, 0}, \gamma, \delta, ) and go to step 1.</td>
</tr>
<tr>
<td>7.</td>
<td>Optimize initial value ( F_{S, 0}, \gamma, \delta, ) and go to step 1.</td>
</tr>
<tr>
<td>8.</td>
<td>Refine mesh if necessary and start again with ( F_{S, 0} := F_S )</td>
</tr>
</tbody>
</table>

### 2.3. Virtual work equation

If we multiply the equilibrium equations by the cartesian element displacements, we can interpret the three scalars \( \hat{A}, A_P \) and \( D_T \) to external work, inner kinetical work and dissipation.

\[
v^T (K_n N + \tilde{K}_i T - F_P)
\]

\[
\hat{v}_n^T K_n^{-1} K_n N + \hat{v}_n^T K_n^{-1} \tilde{K}_i T = v^T F_P
\]

\[
\hat{v}_n^T N = v^T F_P - \hat{v}_n^T K_n^{-1} \tilde{K}_i T
\]

\[
D_T = -\hat{v}_n^T K_n^{-1} \tilde{K}_i T = |v_{0,i}^T| T = v_{0,i}^T \delta T
\]

\[
\hat{A} = A_P + D_T
\]

(39)

(40)

(41)

(42)

(43)

Attention: \( v_{0,i} \) is the tangential term without \( \hat{v}_i \) (if \( \neq 0 \))
Figure 3: 2 elements for a bearing capacity and a slope stability problem

3. Application

3.1. Bearing capacity. We first study the very simple problem according to figure 3. The input data for the matrices are given in the next table.

<table>
<thead>
<tr>
<th>Element 1</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_1 = 250$ kN/m</td>
<td></td>
</tr>
<tr>
<td>$\tan \varphi_1 = 0.8391$</td>
<td>$C_1 = 200$ kN/m</td>
</tr>
<tr>
<td>$l_1 = 0.6$</td>
<td>$n_1 = -0.8$</td>
</tr>
<tr>
<td>$\tan \varphi_2 = 0.8391$</td>
<td>$C_2 = 50$ kN/m</td>
</tr>
<tr>
<td>$l_2 = 0.8$</td>
<td>$n_2 = 0.8$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Element 2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$G_2 = 60$ kN/m</td>
<td></td>
</tr>
<tr>
<td>$\tan \varphi_3 = 0.8391$</td>
<td>$C_3 = 50$ kN/m</td>
</tr>
<tr>
<td>$l_3 = 0.8$</td>
<td>$n_3 = -0.6$</td>
</tr>
<tr>
<td>$l_4 = 0$</td>
<td>$n_4 = 1$</td>
</tr>
</tbody>
</table>

Kinematics and statics (see also appendix for more details). With the assumed vector $\psi_n^p$, which initiates the kinematics

$$\psi_n^p = r^T = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}^T, \dot{\psi}_1 = 0$$

we solve for the absolute

$$\nu^T = \begin{bmatrix} -0.96 & -0.72 & -0.75 & -1.0 \end{bmatrix}^T$$

and the relative displacements
\[ v_i^T = \begin{bmatrix} -1.2 & .35 & -1.25 & .75 \end{bmatrix}^T \]

and derive the signs for the direction of the shear forces.

\[ \delta = \text{sign}(v_i^T) = \begin{bmatrix} -1 & 1 & -1 & 1 \end{bmatrix} \]

Then we build the different matrices and vectors as follows.

\[
K_\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & .29654 & 0 & 0 \\
0 & 1.2713 & 0 & 0
\end{bmatrix}, \quad K_B^{-1} = \begin{bmatrix}
-4.2961 & 1.0021 & 0 & 0 \\
-4.4049 & 2.4088 & 0 & 0 \\
0 & 0 & -3.3722 & 0 \\
0 & 0 & -4.2871 & -1
\end{bmatrix}
\]

\[
F = \begin{bmatrix}
-190 \\
170 \\
0 \\
-20
\end{bmatrix}, \quad f_D = K_B^{-1}F = \begin{bmatrix}
986.62 \\
877.88 \\
0 \\
20
\end{bmatrix}
\]

\[
B_\Delta = K_B^{-1}K_\Delta = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -2.5426 & 0 & 0
\end{bmatrix}
\]

Now we can solve for the unknown effective normal forces with the forward solution

\[
N' = f_D - B_\Delta N' = \begin{bmatrix}
986.62 \\
877.88 \\
0 \\
20
\end{bmatrix} - \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -2.5426 & 0 & 0
\end{bmatrix} \begin{bmatrix}
986.62 \\
877.88 \\
0 \\
2252.1
\end{bmatrix}
\]

and get the limit load to \( P_b = 2252.1 \, \text{kN/m} \).

We may control the results by the virtual work equations.

\[
\hat{A} = \hat{v}^T N' = 2252.1
\]

\[
T = R + C = t^C \hat{N} + C = \begin{bmatrix}
827.87 \\
736.63 \\
736.63 \\
0
\end{bmatrix} + \begin{bmatrix}
200 \\
50 \\
50 \\
0
\end{bmatrix} = \begin{bmatrix}
1027.9 \\
786.63 \\
786.63 \\
0
\end{bmatrix}
\]

\[
D_i = D_R + D_C = |v_{0,i}^T| (R + C) = |v_{0,i}^T| T = 2172.1 + 320 = 2492.1
\]

\[
A_p = v^T F_p = -240
\]

\[
\hat{A} = 2252.1 \equiv A_p + D_i = -240 + 2492.1 \text{ (correct)}.
\]
3.2. **Slope stability.** In principle we have to solve the implicit system of equations \( v^T \mathbf{N}(F_S) = 0 \) for \( F_S \). We start with the guess

\[
F_{S,0} = 1, F_S = F_{S,0} = 1
\]

and compute

\[
\mathbf{D} \mathbf{K}_{0, B}^{\text{adj}} = \begin{pmatrix}
-2.0276 & 2.7034 & 0 & 0 \\
-2.7034 & -2.0276 & 0 & 0 \\
0 & 0 & -3.3722 & 0 \\
0 & 0 & -2.0233 & -2.6978
\end{pmatrix}
\]

remark: use \( \mathbf{D} \) and not \( \mathbf{D}_n \)

\[
\mathbf{D} \mathbf{K}_{1, B}^{\text{adj}} = F_S \left( \mathbf{K}_{B}^{-1} - \mathbf{D} \mathbf{K}_{0, B}^{\text{adj}} \right) = \begin{pmatrix}
-2.2685 & -1.7013 & 0 & 0 \\
-1.7015 & 2.2685 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -2.2638 & 1.6978
\end{pmatrix}
\]

\[
\mathbf{F} = \mathbf{F}_0 + \frac{1}{F_S} \mathbf{F}_1
\]

\[
\mathbf{F}_0 = \begin{pmatrix}
0 \\
250 \\
0 \\
60
\end{pmatrix}, \mathbf{F}_1 = \begin{pmatrix}
-190 \\
-80 \\
0 \\
-80
\end{pmatrix}
\]

\[
f_{0,D} = \mathbf{D} \mathbf{K}_{0, B}^{\text{adj}} \mathbf{F}_0 = \begin{pmatrix}
675.85 \\
-506.9 \\
0 \\
-161.87
\end{pmatrix}
\]

\[
f_{1,D} = \mathbf{D} \mathbf{K}_{0, B}^{\text{adj}} \mathbf{F}_1 + \mathbf{D} \mathbf{K}_{1, B}^{\text{adj}} \mathbf{F}_0 = \begin{pmatrix}
-256.35 \\
1243.0 \\
0 \\
317.69
\end{pmatrix}
\]

\[
f_{2,D} = \mathbf{D} \mathbf{K}_{1, B}^{\text{adj}} \mathbf{F}_1 = \begin{pmatrix}
567.12 \\
141.81 \\
0 \\
-135.82
\end{pmatrix}
\]

and solve three times with the update solution for \( f_{i,D}^* = f_{i,D} - \mathbf{B} \Delta f_{i,D} \), \( i = 0,1,2 \)

\[
f_{0,D}^* = \begin{pmatrix}
675.85 \\
-506.9 \\
-506.9 \\
-1450.7
\end{pmatrix}, f_{1,D}^* = \begin{pmatrix}
-256.35 \\
1243.0 \\
3475.1
\end{pmatrix}, f_{2,D}^* = \begin{pmatrix}
567.12 \\
141.81 \\
224.75
\end{pmatrix}
\]
\[
P_b = \psi_n^T N = 0
\]
\[
\text{Norm} = \psi_n^T f_b = -1450.7
\]
\[
a_1 = \frac{\psi_n^T f_0}{\text{Norm}} = -2.3975
\]
\[
a_0 = \frac{\psi_n^T f_0}{\text{Norm}} = -0.1549
\]
\[
F_S = \frac{a_1}{2} + \sqrt{\frac{a_1^2}{4} - a_0} = 2.46046
\]

Further iterations show the excellent convergence of the iteration technique developed above.

<table>
<thead>
<tr>
<th>Nr.</th>
<th>result $F_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$F_{S0} = 1$</td>
</tr>
<tr>
<td>1</td>
<td>2.46046</td>
</tr>
<tr>
<td>2</td>
<td>2.30453</td>
</tr>
<tr>
<td>3</td>
<td>2.30005</td>
</tr>
<tr>
<td>4</td>
<td>2.30003</td>
</tr>
</tbody>
</table>

We may also control the results by the virtual work equation

\[
K = K_0 + \frac{1}{F_S} K_1 = \begin{bmatrix}
-31628 & -58721 & 0 & 0 \\
10128 & -88372 & 0 & 0 \\
0 & 58721 & -58721 & 0 \\
0 & 88372 & 88372 & -1
\end{bmatrix},
\]

\[
K^{-1} = \begin{bmatrix}
-1.0109 & 67169 & 0 & 0 \\
-1.1585 & -36178 & 0 & 0 \\
-1.1585 & -36178 & -1.703 & 0 \\
-2.0476 & -63943 & -1.5049 & -1
\end{bmatrix},
\]

\[
N'_\text{mob} = K^{-1} \left( F_0 + \frac{1}{F_S} F_1 \right) = \begin{bmatrix}
226.39 \\
14.818 \\
14.818 \\
0
\end{bmatrix}
\]

\[
\hat{A} = \psi_n^T N'_\text{mob} = 0
\]

\[
T_{\text{mob}} = R_{\text{mob}} + C_{\text{mob}} = \begin{bmatrix}
80.288 \\
5.2551 \\
5.2551 \\
0
\end{bmatrix} + \begin{bmatrix}
84.53 \\
21.132 \\
21.132 \\
0
\end{bmatrix} = \begin{bmatrix}
164.82 \\
26.387 \\
26.387 \\
0
\end{bmatrix}
\]

\[
D_{1,\text{mob}} = D_{h,\text{mob}} + D_{C,\text{mob}} = |\psi_{0,1}^T| (R_{\text{mob}} + C_{\text{mob}}) = 104.75 + 135.25 = 240.0
\]

\[
A_P = -240
\]

\[
\hat{A} = 0 = A_P + D_{1,\text{mob}} = -240 + 240.0 = 0 \quad \text{(correct)}.
\]
failure mechanism with 8 elements

$F_S = 1.9935$

$\phi = 40^\circ$

$c = 20 \, \text{kN/m}$

$\gamma = 20 \, \text{kN/m}$

$\beta = 50.91^\circ$

$H = 8.0 \, \text{m}$

Figure 4: Kinematics for the optimized geometry

Until now we have not optimized neither the geometry nor the mesh. Figure 4 shows the kinematics for a system of 8 elements and the next table shows some more results.

<table>
<thead>
<tr>
<th>Elements</th>
<th>Limit Load $P_h , [\text{kN/m}]$</th>
<th>$F_S$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 elements</td>
<td>1213 ($B = 3, m$)</td>
<td>2.0740</td>
</tr>
<tr>
<td>8 elements</td>
<td>1130 ($B = 3, m$)</td>
<td>1.9935</td>
</tr>
</tbody>
</table>

So far we have not studied the possible influence of a “horizontal” subdivision of the failure mesh. To do this, we have to change some data, because in the above example the high value for the cohesion dominates the results.

“Horizontally” subdivided mesh.

We reduce the cohesion from 20 to 4 $\, \text{kN/m}$ and assume a footing on top of the surface with a constant load $P = 100 \, \text{kN/m}$ in case of the slope stability problem. We see from the next table

<table>
<thead>
<tr>
<th>Subdivision</th>
<th>Limit Load $P , [\text{kN/m}]$</th>
<th>$F_S$ (load $P = 100 , \text{kN/m}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>3 elements (only “vertical” subdivision)</td>
<td>285.8</td>
<td>1.1195</td>
</tr>
<tr>
<td>6 elements (only “vertical” subdivision)</td>
<td>272.2</td>
<td>1.1109</td>
</tr>
<tr>
<td>12 elements (“horizontal” subdivision)</td>
<td>270.1</td>
<td>1.1089 (see figure 5)</td>
</tr>
</tbody>
</table>

the tiny effect of the “horizontal” subdivision.

4. Conclusion

Besides the simplicity of the different solutions it could be proved in examples

- the rapid and stable convergency of the proposed iteration scheme for the slope stability problem

- the dominant influence of the “vertical” subdivision in contrast to the negligible influence of a “horizontal” subdivision.
5. APPENDIX

5.1. Kinematics (no dilatancy). The kinematics for a single boundary may be written as follows

\[
\begin{align*}
\vec{v}_{e/f} &= \vec{v}_e - \vec{v}_e \\
\vec{v}_{n,e/f} &= \left[\vec{v}_{e/f}\right] = -\delta n |\vec{v}_{e/f}| = -n\vec{v}_{e,f}, \vec{v}_{z,e/f} = \bar{n} |\vec{v}_{e/f}| = \delta l |\vec{v}_{e/f}| = l\vec{v}_{e,f} \\
\vec{l} &= \frac{\vec{v}_{z,e/f}}{|\vec{v}_{e/f}|} \bar{n} = \frac{\vec{v}_{z,e/f}}{|\vec{v}_{e/f}|} \delta = \frac{\vec{l}}{n} = \vec{n} \\
\vec{v}_{\text{normal},e/f} &= \vec{v}_{\text{tangential},e/f} = \vec{v}_{e/f} - n_{e/f} \vec{v}_{z,e/f} + n_{e/f} \vec{v}_{z,e/f} = 0, \hat{v}_n = \text{positive for passive problems} \\
\vec{v}_l &= -(l_{f/f} \hat{v}_x + n_{f/f} \hat{v}_z) \\
\vec{v}_{l,e/f} &= \delta |\vec{v}_{e/f}| (n^2 + l^2) = \delta |\vec{v}_{e/f}| = \vec{v}_{e/f}, \vec{v}_{l,e/f} = + \text{if rotates counterclockwise in } f \\
\vec{v}_{l,f/f} &= \vec{v}_{l,l} + \hat{v}_l \\
\vec{u} &= +(-n_{f/f} \hat{v}_x + l_{f/f} \hat{v}_z) = +n_{f/f} \hat{v}_x - l_{f/f} \hat{v}_z \\
\end{align*}
\]

Effective solution for the kinematics (according to the statics)

\[
\begin{align*}
-K_v \vec{v} &= \vec{v}_n, -K_v \vec{K}_n = K_{n,B} + K_{n,\Delta} \\
K_{n,B}^T \vec{v} + K_{n,\Delta} \vec{v} &= \vec{v}_n, K_{n,B}^T \vec{v} = \vec{v}_n - K_{n,\Delta} \vec{v} \\
\vec{v} &= (K_{n,B}^T)^{-1} (\vec{v}_n - K_{n,\Delta} \vec{v}) \\
\vec{v}^T &= \vec{v}_n^T K_{n,B}^{-1} - \vec{v}_n^T K_{n,\Delta} K_{n,B}^{-1} \text{ (update solution)} \\
K_{n,B}^{-1} &= D_n K_{n,B}^{adj} \\
\end{align*}
\]
$$v^T_i = -v^TK_1 + \tilde{v}_i^T$$

5.2. Statics. If $N$ denotes the normal force, $T$ the tangential and $S$ the resultant of both, the statics for a single boundary may be written as

$$N_{x_0} = -TN_x, N_{z_0} = -nN$$
$$T_{x_0} = -\tilde{t}N, T_{z_0} = -\tilde{n}N$$
$$S_x = -(TN_0 + \tilde{t}N_0), S_z = -(nN + \tilde{n}N_0)$$
$$\tilde{l} = -\delta n, \tilde{n} = \delta t$$
$$\tilde{l} = -(l + \tilde{t}\tan\varphi), \tilde{n} = -(n + \tilde{n}\tan\varphi)$$

which leads together with the equilibrium equations for a single element to

$$\sum S_x + P_x + B_x + A_x = 0, \sum S_z + P_z + B_z + A_z = 0$$
$$\sum S_x = -(P_x + B_x + A_x) = F_{P_x}$$
$$\sum S_z = -(P_z + B_z + A_z) = F_{P_z}$$

where $P$ is a single force, $B$ a body force and $A$ a surface force.

5.3. Inverse according to Householder [4].

$$\left(A + UV^T\right)^{-1} = A^{-1} - A^{-1}(I + V^TA^{-1}U)^{-1}V^TA^{-1}$$
$$\left(I +UU^T\right)^{-1} = I - U(I + V^TU)^{-1}V^T$$

yields

$$W = (I + K_B^{-1}K_\Delta)^{-1} = I - K_B^{-1}(I + K_\Delta K_B^{-1})^{-1}K_\Delta$$
$$(I + K_\Delta K_B^{-1})^{-1} = I - K_\Delta(I + K_B^{-1}K_\Delta)^{-1}K_B^{-1}$$
$$W = I - K_B^{-1}\left[I - K_\Delta(I + K_B^{-1}K_\Delta)^{-1}K_B^{-1}\right]K_\Delta$$
$$W = I - K_B^{-1}K_\Delta + K_B^{-1}K_\Delta WK_B^{-1}K_\Delta$$
$$B_\Delta = K_B^{-1}K_\Delta$$ (abbreviation)
$$W = I - B_\Delta + B_\Delta WB_\Delta$$

and

$$W = I - B_\Delta + B_\Delta(I - B_\Delta + B_\Delta WB_\Delta)B_\Delta$$
$$W = (I + K_B^{-1}K_\Delta)^{-1} = I + \sum_{i=1}^{\nu} (-1)^i B_\Delta^{i}, B_\Delta^{i+1} = 0$$
$$K_\Delta^{-1} = \left(I + \sum_{i=1}^{\nu} (-1)^i B_\Delta^{i}\right)K_B^{-1}, B_\Delta^{i+1} = 0$$

As an alternative we can derive $(I + B_\Delta)^{-1} = I + \sum_{i=1}^{\nu} (-1)^i B_\Delta^{i}$ from the Taylor-series $(1 + x)^{-1} = -\frac{1}{x} + \frac{1}{x^2} - \frac{1}{x^3} + \frac{1}{x^4} - \ldots$. But this is not a common technique for matrices.
It is remarked, that the result is similar to the BFGS formulation given by Matthies and Strang [5]. But in FEM $B_{\Delta}$ is a matrix with small elements and the resulting formula is an approximation. Here $B_{\Delta}$ is a sparse lower matrix and the formula is exact, because of $B_{\Delta}^{-1} = 0$, where $\nu = n + m - 2$ is finite.

W in case of $m = 1$. For this most important case in practice we can derive the element components of matrix $W$ from the generalisation of the following example with $m = 1, n = 2, \nu = m + n - 2 = n - 1 = 1$.

$$K_B = \begin{pmatrix}
\tilde{l}_1 & \tilde{l}_2 & 0 & 0 \\
\tilde{n}_1 & \tilde{n}_2 & 0 & 0 \\
0 & 0 & \tilde{l}_3 & \tilde{l}_4 \\
0 & 0 & \tilde{n}_3 & \tilde{n}_4
\end{pmatrix}, K_{\Delta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -\tilde{l}_2 & 0 & 0 \\
0 & -\tilde{n}_2 & 0 & 0
\end{pmatrix}$$

$$-B_{\Delta} = -K_B^{-1}K_{\Delta} = -DK_B^{\text{adj}}K_{\Delta} = DQ_{\Delta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \tilde{l}_2\tilde{n}_4 - \tilde{n}_2\tilde{l}_4 & \tilde{l}_2\tilde{n}_4 - \tilde{n}_2\tilde{l}_4 & 0 \\
0 & \tilde{l}_2\tilde{n}_4 - \tilde{n}_2\tilde{l}_4 & \tilde{l}_2\tilde{n}_4 - \tilde{n}_2\tilde{l}_4 & 0
\end{pmatrix}$$

$$Q_{\Delta} = -K_B^{\text{adj}}K_{\Delta}$$

$$W = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & w_{3,2} & \frac{q_{3,2}}{D_x} & 1 \\
0 & w_{4,2} & \frac{q_{4,2}}{D_x} & 0 & 1
\end{pmatrix}$$

In general

$$w_{i,j} = 0 \text{ if } i - j < 0 \text{ or } i = \text{ odd and } j \neq i$$

$$w_{i,i} = 1$$

$$w_{2i-1,2i-2} = \frac{\tilde{l}_{2i-2}\tilde{n}_{2i-2} - \tilde{n}_{2i-2}\tilde{l}_{2i-2}}{l_{2i-1}\tilde{n}_{2i-1} - \tilde{n}_{2i-1}\tilde{l}_{2i-1}} = \frac{q_{2i-1,2i-2}}{D_i}$$

$$w_{2i,2i-2} = \frac{-\tilde{l}_{2i-2}\tilde{n}_{2i-1} - \tilde{n}_{2i-2}\tilde{l}_{2i-1}}{l_{2i-1}\tilde{n}_{2i-1} - \tilde{n}_{2i-1}\tilde{l}_{2i-1}} = \frac{q_{2i,2i-2}}{D_i}$$

$$w_{i,2j} = w_{i,j+2}w_{2j+2,j} \text{ if } i - 2j > 2$$

5.4. Application.

Kinematics for the first example.

$$K_{n,B} = \begin{pmatrix}
-6 & -8 & 0 & 0 \\
8 & -6 & 0 & 0 \\
0 & 0 & -8 & 0 \\
0 & 0 & .6 & -1
\end{pmatrix}, K_{n,\Delta} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & .8 & 0 & 0 \\
0 & .6 & 0 & 0
\end{pmatrix}$$

$$K_{n,B}^{\text{adj}} = \begin{pmatrix}
-6 & .8 & 0 & 0 \\
-8 & -6 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -6 & -8
\end{pmatrix}, D_n = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1.25 & 0 \\
0 & 0 & 0 & 1.25
\end{pmatrix}$$
\[ K_{n,\mu}^{-1} = D_n K_{n,\mu}^{\psi\psi} = \begin{pmatrix} -6 & .8 & 0 & 0 \\ -8 & -6 & 0 & 0 \\ 0 & 0 & -1.25 & 0 \\ 0 & 0 & -1.75 & -1.0 \end{pmatrix} \]

\[ K_{n,\Delta K_{n,\mu}^{-1}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.64 & -0.48 & 0 & 0 \\ -0.48 & -3.60 & 0 & 0 \end{pmatrix} \]

\[ v^T = \tilde{v}_n^T K_{n,\mu}^{-1} - v^T K_{n,\Delta K_{n,\mu}^{-1}} = \begin{pmatrix} -0.96 & -0.72 & -1.75 & -1.0 \end{pmatrix}^T \]

\[ v_i^T = -v^T K_i + \tilde{v}_i^T = \begin{pmatrix} -1.2 & 0.35 & -1.25 & 0.75 \end{pmatrix}^T \]

\[ \delta = \text{sign}(v_i^T) = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \]

Statics.

\[ K_0 \equiv K_n = \begin{pmatrix} -l_1 & -l_2 \\ -n_1 & -n_2 \\ l_2 & -l_3 & -l_4 \\ n_2 & -n_3 & -n_4 \end{pmatrix} = \begin{pmatrix} -0.6 & -0.8 \\ 0.8 & -0.6 \\ 0.8 & -0.8 & 0 \\ 0.6 & 0.6 & -1 \end{pmatrix} \]

\[ K_i = \begin{pmatrix} n_1 & n_2 \\ -l_1 & -l_2 \\ -n_2 & n_3 & n_4 \\ l_2 & -l_3 & -l_4 \end{pmatrix} = \begin{pmatrix} -0.8 & 0.6 \\ -0.8 & -0.6 \\ -0.8 & -0.8 & 1 \\ 0.8 & 0 \end{pmatrix} \]

\[ \delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \delta_4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \]

\[ \bar{K}_i = K_i \delta = \begin{pmatrix} .8 & .6 & 0 & 0 \\ .6 & -0.8 & 0 & 0 \\ 0 & -.6 & .6 & 1 \\ 0 & 0.6 & .8 & 0 \end{pmatrix} \]

\[ t_\psi = \begin{pmatrix} \tan \varphi_1 \\ \tan \varphi_2 \\ \tan \varphi_3 \\ 0 \end{pmatrix} = \text{diag} \{ .8391, .8391, .8391, 0 \} \]
\[
K_R = \bar{K}_1 t_\varphi = \\
\begin{pmatrix}
.67128 & .50346 & 0 & 0 \\
.50346 & -.67128 & 0 & 0 \\
0 & -.50346 & .50346 & 0 \\
0 & .67128 & .67128 & 0 \\
\end{pmatrix}
\]

\[
K = K_0 + K_R = \\
\begin{pmatrix}
\tilde{i}_1 & \tilde{i}_2 & \tilde{i}_3 & \tilde{i}_4 \\
\tilde{n}_1 & \tilde{n}_2 & \tilde{n}_3 & \tilde{n}_4 \\
\end{pmatrix}
\]

\[
= \\
\begin{pmatrix}
.07128 & -.2965 & 0 & 0 \\
1.3035 & -1.2713 & 0 & 0 \\
0 & 0 & -.2965 & 0 \\
0 & 0 & 1.2713 & -1 \\
\end{pmatrix}
\]

\[
K_B = \\
\begin{pmatrix}
\tilde{n}_2 & -\tilde{i}_2 & \tilde{n}_4 & -\tilde{i}_4 \\
\tilde{i}_1 & \tilde{i}_3 & \tilde{n}_3 & \tilde{i}_4 \\
\end{pmatrix}
\]

\[
= \\
\begin{pmatrix}
0 & 0 & -.2965 & 0 \\
0 & 0 & 1.2713 & -1 \\
0 & 0 & 0 & 3.3722 \\
0 & 0 & 0 & 3.3722 \\
\end{pmatrix}
\]

\[
D_1 = \det \begin{vmatrix} 0.0712 & -.2965 \\ 1.3035 & -1.2712 \end{vmatrix} = .29592 \\
\frac{1}{D_1} = 3.3793
\]

\[
D_2 = \det \begin{vmatrix} -.2965 & 0 \\ 1.2712 & -1 \end{vmatrix} = .29654 \\
\frac{1}{D_2} = 3.3722
\]

\[
D = \\
\begin{pmatrix}
3.3793 & 0 & 0 & 0 \\
0 & 3.3793 & 0 & 0 \\
0 & 0 & 3.3722 & 0 \\
0 & 0 & 0 & 3.3722 \\
\end{pmatrix}
\]

\[
K_B^{\sigma\varphi} = \\
\begin{pmatrix}
\tilde{n}_2 & -\tilde{i}_2 & \tilde{n}_4 & -\tilde{i}_4 \\
\tilde{i}_1 & \tilde{i}_3 & \tilde{n}_3 & \tilde{i}_4 \\
\end{pmatrix}
\]

\[
= \\
\begin{pmatrix}
-1.2713 & .29654 & 0 & 0 \\
-1.3035 & .07128 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & -1.2713 & -.29654 \\
\end{pmatrix}
\]

\[
K_B^{-1} = D K_B^{\sigma\varphi} = \\
\begin{pmatrix}
-4.2961 & 1.0021 & 0 & 0 \\
-4.4049 & .24088 & 0 & 0 \\
0 & 0 & -3.3722 & 0 \\
0 & 0 & -4.2871 & -1 \\
\end{pmatrix}
\]

\[
B_\Delta = K_B^{-1} K_\Delta = \\
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & -2.5426 & 0 & 0 \\
\end{pmatrix}
\]
\[ \mathbf{F}_C = -\mathbf{K}_1 \mathbf{C} = - \begin{pmatrix} .8 & .6 \\ .6 & -.8 \\ .8 & .6 \end{pmatrix} \begin{pmatrix} 200 \\ 50 \\ 50 \end{pmatrix} = \begin{pmatrix} -190 \\ -80 \\ 0 \end{pmatrix} \]

\[ \mathbf{F}_0 = -(\mathbf{P} + \mathbf{B} + \mathbf{A} + \mathbf{K}_n \mathbf{U}) = - \begin{pmatrix} 0 \\ -250 \\ 0 \end{pmatrix} \]

\[ \mathbf{F} = \mathbf{F}_0 + \mathbf{F}_C = - \begin{pmatrix} 0 \\ -250 \\ 0 \end{pmatrix} + \begin{pmatrix} -190 \\ -80 \\ 0 \end{pmatrix} = \begin{pmatrix} -190 \\ 170 \\ 0 \end{pmatrix} \]

**References**


